

Spinorspaces in discrete Clifford analysis

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Abstract

In this paper we work in the ‘split’ discrete Clifford analysis setting, i.e. the m -dimensional function theory concerning null-functions, defined on the grid \mathbb{Z}^m , of the discrete Dirac operator ∂ , involving both forward and backward differences, which factorizes the (discrete) Star-Laplacian ($\Delta^* = \partial^2$). We show how the space \mathcal{M}_k of discrete homogeneous spherical monogenics of degree k , is decomposable into 2^{2m-n} isomorphic irreducible representations with highest weight $(k + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ in the odd-dimensional case and two times 2^{2m-n} isomorphic irreducible representations with highest weight $(k)_+ = (k + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2})$ resp. $(k)_- = (k + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$ in the even dimensional case.

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1 Introduction

In classical Clifford analysis, the infinitesimal ‘rotations’ are given by the angular momentum operators $L_{a,b} = x_a \partial_{x_b} - x_b \partial_{x_a}$. These operators satisfy the commutation relations

$$[L_{a,b}, L_{c,d}] = \delta_{b,c} L_{a,d} - \delta_{b,d} L_{a,c} - \delta_{a,c} L_{b,d} + \delta_{a,d} L_{b,c},$$

which are exactly the defining relations of the special orthogonal Lie algebra $\mathfrak{so}(m)$ and they form endomorphisms of the space $\mathcal{H}_k(m, \mathbb{C})$ of scalar-valued harmonic homogeneous polynomials, thus transforming the latter in an (irreducible) $\mathfrak{so}(m, \mathbb{C})$ -representation. To establish $\mathcal{M}_k(m, \mathbb{S})$, i.e. the spinor-valued homogeneous monogenics of degree k , classically as $\mathfrak{so}(m, \mathbb{C})$ -representation, the following operators are considered

$$dR(e_{a,b}) : \mathcal{M}_k(m, \mathbb{S}) \rightarrow \mathcal{M}_k(m, \mathbb{S}), \quad M_k \mapsto \left(L_{a,b} + \frac{1}{2} e_a e_b \right) M_k.$$

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These operators are endomorphisms of the space of spinor-valued k -homogeneous polynomials which also satisfy the defining relations of $\mathfrak{so}(m, \mathbb{C})$:

$$[dR(e_{a,b}), dR(e_{c,d})] = \delta_{b,c} dR(e_{a,d}) - \delta_{b,d} dR(e_{a,c}) - \delta_{a,c} dR(e_{b,d}) + \delta_{a,d} dR(e_{b,c}).$$

In [3], we developed similar operators in the discrete Clifford analysis setting: the angular momentum operators are discrete operators $L_{a,b} = \xi_a \partial_b + \xi_b \partial_a$, $a \neq b$. For $a = b$, we define $L_{aa} = 0$. Then the operators $\Omega_{a,b}$, acting on discrete functions f as $\Omega_{a,b} f = L_{a,b} f e_b e_a$, satisfy the defining relations of the special lie algebra $\mathfrak{so}(m)$:

$$[\Omega_{a,b}, \Omega_{c,d}] = \delta_{b,c} \Omega_{a,d} - \delta_{b,d} \Omega_{a,c} - \delta_{a,c} \Omega_{b,d} + \delta_{a,d} \Omega_{b,c}.$$

Furthermore, they are endomorphisms of the space \mathcal{H}_k of Clifford-algebra valued homogeneous harmonics of degree k , since $\Omega_{a,b}$ commutes with $\mathfrak{sl}_2 = \{\Delta, \xi^2, \mathbb{E} + \frac{m}{2}\}$, $\forall (a, b)$. In [4], we showed that \mathcal{H}_k is the sum of 2^{2m} isomorphic copies of the irreducible representation of $\mathfrak{so}(m, \mathbb{C})$ with highest weight $(k, 0, \dots, 0)$.

The discrete Dirac operator ∂ is however not invariant under the operators $\Omega_{a,b}$, hence \mathcal{M}_k cannot be expressed as $\mathfrak{so}(m, \mathbb{C})$ -representation by means of these operators. Therefore, we considered in [3] the operators $L_{a,b} - \frac{1}{2}$ and the four-vector $V_{a,b} = e_a e_b e_a^\perp e_b^\perp = -e_a^\perp e_a e_b^\perp e_b$. Let the operator $dR(e_{a,b})$, $a \neq b$, act on discrete functions f as

$$dR(e_{a,b}) f = V_{a,b} \left(L_{a,b} - \frac{1}{2} \right) f e_a^\perp e_b^\perp.$$

For $a = b$, we defined $dR(e_{a,a}) = 0$. The operators $dR(e_{a,b})$ satisfy the defining relations of the special lie algebra $\mathfrak{so}(m)$:

$$[dR(e_{a,b}), dR(e_{c,d})] = \delta_{b,c} dR(e_{a,d}) - \delta_{b,d} dR(e_{a,c}) - \delta_{a,c} dR(e_{b,d}) + \delta_{a,d} dR(e_{b,c}),$$

and commute with $\mathfrak{osp}(1|2) = \{\partial, \xi, \mathbb{E} + \frac{m}{2}\}$ which makes them endomorphisms of the space of k -homogeneous discrete monogenic polynomials. As such, the space \mathcal{M}_k of k -homogeneous Clifford-valued monogenic polynomials is a reducible $\mathfrak{so}(m, \mathbb{C})$ -representation. In [3], it was already suggested that \mathcal{M}_k can be decomposed into irreducible parts of highest weight $(k + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ resp. $(k + \frac{1}{2}, \frac{1}{2}, \dots, -\frac{1}{2})$, but this was left as open conjecture. In the following sections, we will show how this decomposition is done exactly.

2 Preliminaries

Let \mathbb{R}^m be the m -dimensional Euclidian space with orthonormal basis e_j , $j = 1, \dots, m$ and consider the Clifford algebra $\mathbb{R}_{m,0}$ over \mathbb{R}^m . Passing to the so-called ‘split’ discrete setting [5, 1], we imbed the Clifford algebra $\mathbb{R}_{m,0}$ into the bigger complex one $\mathbb{C}_{2m,0}$, the underlying

vector space of which has twice the dimension, and introduce forward and backward basis elements \mathbf{e}_j^\pm satisfying the following anti-commutator rules:

$$\left\{ \mathbf{e}_j^-, \mathbf{e}_\ell^- \right\} = \left\{ \mathbf{e}_j^+, \mathbf{e}_\ell^+ \right\} = 0, \quad \left\{ \mathbf{e}_j^+, \mathbf{e}_\ell^- \right\} = \delta_{j\ell}, \quad j, \ell = 1, \dots, m.$$

The connection to the original basis \mathbf{e}_j is given by $\mathbf{e}_j^+ + \mathbf{e}_j^- = \mathbf{e}_j$, $j = 1, \dots, m$. This implies $e_j^2 = 1$, in contrast to the usual Clifford setting where traditionally $e_j^2 = -1$ is chosen. We will often denote $\mathbf{e}_j^+ \wedge \mathbf{e}_j^- = \mathbf{e}_j^+ \mathbf{e}_j^- - \mathbf{e}_j^- \mathbf{e}_j^+$, $j = 1, \dots, m$.

Now consider the standard equidistant lattice \mathbb{Z}^m ; the coordinates of a Clifford vector \underline{x} will thus only take integer values. We construct a discrete Dirac operator factorizing the discrete Laplacian, using both forward and backward differences Δ_j^\pm , $j = 1, \dots, m$, acting on Clifford-valued functions f as follows:

$$\Delta_j^+[f](\cdot) = f(\cdot + \mathbf{e}_j) - f(\cdot), \quad \Delta_j^-[f](\cdot) = f(\cdot) - f(\cdot - \mathbf{e}_j).$$

With respect to the \mathbb{Z}^m -grid, the usual definition of the discrete Laplacian in $\underline{x} \in \mathbb{Z}^m$ is

$$\Delta^*[f](\underline{x}) = \sum_{j=1}^m \Delta_j^+ \Delta_j^- [f] = \sum_{j=1}^m (f(\underline{x} + \mathbf{e}_j) + f(\underline{x} - \mathbf{e}_j)) - 2m f(\underline{x}).$$

This operator is also known as ‘‘Star Laplacian’’; we will from now on simply write Δ . An appropriate definition of a discrete Dirac operator ∂ factorizing Δ , i.e. satisfying $\partial^2 = \Delta$, is obtained by combining the forward and backward basis elements with the corresponding forward and backward differences, more precisely

$$\partial = \sum_{j=1}^m \left(\mathbf{e}_j^+ \Delta_j^+ + \mathbf{e}_j^- \Delta_j^- \right).$$

In order to receive an analogue of the classical Weyl relations $\partial_{x_j} x_k - x_k \partial_{x_j} = \delta_{jk}$, the co-ordinate vector variable operators $\xi_j = \mathbf{e}_j^+ X_j^- + \mathbf{e}_j^- X_j^+$ are defined by their interaction with the corresponding co-ordinate operators $\partial_j = \mathbf{e}_j^+ \Delta_j^+ + \mathbf{e}_j^- \Delta_j^-$, according to the skew Weyl relations, cf. [1]

$$\partial_j \xi_j - \xi_j \partial_j = 1, \quad j = 1, \dots, m,$$

which imply that $\partial_j \xi_j^k[1] = k \xi_j^{k-1}[1]$. The operators ξ_j and ∂_j furthermore satisfy the following anti-commutator relations:

$$\{\xi_j, \xi_k\} = \{\partial_j, \partial_k\} = \{\partial_j, \xi_k\} = 0, \quad j \neq k, \quad j, k = 1, \dots, m$$

implying that $\partial_\ell \xi_j^k[1] = 0$, $j \neq \ell$.

The natural powers $\xi_j^k[1]$ of the operator ξ_j acting on the ground state 1 are the basic discrete k -homogeneous polynomials of degree k in the variable x_j , i.e. $\mathbb{E} \xi_j^k[1] = k \xi_j^{k-1}[1]$,

where $\mathbb{E} = \sum_{j=1}^m \xi_j \partial_j$ is the discrete Euler operator. They constitute a basis for all discrete polynomials. Explicit formulas for $\xi_j^k[1]$ are given for example in [1, 2]; furthermore $\xi_j^k[1](x_j) = 0$ if $k \geq 2|x_j| + 1$.

A discrete function is discrete harmonic (resp. left discrete monogenic) in a domain $\Omega \subset \mathbb{Z}^m$ if $\Delta f(\underline{x}) = 0$ (resp. $\partial f(\underline{x}) = 0$), for all $\underline{x} \in \Omega$. The space of discrete harmonic (resp. monogenic) homogeneous polynomials of degree k is denoted \mathcal{H}_k (resp. \mathcal{M}_k), while the space of all discrete harmonic (resp. monogenic) homogeneous polynomials is denoted \mathcal{H} (resp. \mathcal{M}). It is clear that

$$\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k, \quad \mathcal{M} = \bigoplus_{k=0}^{\infty} \mathcal{M}_k.$$

The respective dimensions over the discrete Clifford algebra are

$$\dim(\mathcal{H}_k) = \binom{k+m-1}{k} - \binom{k+m-3}{k}, \quad \dim(\mathcal{M}_k) = \binom{k+m-2}{k}.$$

3 Orthogonal Lie algebras

We will start by briefly introducing the orthogonal Lie algebra $\mathfrak{so}(m, \mathbb{C})$; a detailed description can be found for example in [6]. The orthogonal Lie algebra $\mathfrak{so}(m, \mathbb{C})$ is generated in even dimension $m = 2n$ by $\binom{m}{2}$ basis elements $H_a, X_{a,b}, Y_{a,b}$ and $Z_{a,b}$ ($1 \leq a, b \leq n$) and in odd dimension $m = 2n + 1$ these basis elements are extended to a full basis of $\mathfrak{so}(m, \mathbb{C})$ by $2n$ extra elements U_a and V_a , $1 \leq a \leq n$:

$$\begin{aligned} \mathfrak{so}(2n, \mathbb{C}) &= \text{span}_{\mathbb{C}} \{H_a, X_{a,b}, Y_{a,b}, Z_{a,b}, 1 \leq a, b \leq n, a \neq b\}, \\ \mathfrak{so}(2n+1, \mathbb{C}) &= \text{span}_{\mathbb{C}} \{H_a, X_{a,b}, Y_{a,b}, Z_{a,b}, U_a, V_a, 1 \leq a, b \leq n, a \neq b\}. \end{aligned}$$

The Cartan subalgebra can be chosen as

$$\mathfrak{h} = \{H_a, 1 \leq a \leq n\},$$

independently of the parity of the dimension, i.e. $\mathfrak{so}(2n, \mathbb{C})$ and $\mathfrak{so}(2n+1, \mathbb{C})$ are both Lie algebras of rank n . The roots of $\mathfrak{so}(m, \mathbb{C})$ (see also [?]) are determined by considering the adjoint representation ($1 \leq a, b, c, d \leq n$):

$$\begin{aligned} [H_c, Y_{a,b}] &= (\delta_{ca} + \delta_{cb}) Y_{a,b} = ((L_a + L_b)(H_c)) Y_{a,b}, \\ [H_c, X_{a,b}] &= (\delta_{ca} - \delta_{cb}) X_{a,b} = ((L_a - L_b)(H_c)) X_{a,b}, \\ [H_c, Z_{a,b}] &= -(\delta_{ca} + \delta_{cb}) Z_{a,b} = ((-L_a - L_b)(H_c)) Z_{a,b}, \\ [H_c, U_a] &= \delta_{ca} U_a = (L_a(H_c)) U_a, \\ [H_c, V_a] &= -\delta_{ca} U_a = (-L_a(H_c)) U_a. \end{aligned}$$

Note in particular that the Cartan subalgebra elements H_a can be found by means of the commutator of a positive root with a negative root of the same index:

$$[Y_{a,b}, Z_{a,b}] = -H_a - H_b, \quad [X_{a,b}, X_{b,a}] = H_a - H_b.$$

We thus deduce the following roots and root vectors. Here $\{L_a, 1 \leq a \leq n\}$ is a basis of the dual vector space \mathfrak{h}^* of the Cartan subalgebra \mathfrak{h} , i.e. $L_a(H_b) = \delta_{a,b}$.

$m = 2n$		$m = 2n + 1$	
root	root vector	root	root vector
$L_a - L_b$	$X_{a,b}$	$L_a - L_b$	$X_{a,b}$
$L_a + L_b$	$Y_{a,b}$	$L_a + L_b$	$Y_{a,b}$
$-L_a - L_b$	$Z_{a,b}$	$-L_a - L_b$	$Z_{a,b}$
		L_a	U_a
		$-L_a$	V_a

By the usual convention, we choose the positive roots in even dimension to be

$$\{L_a + L_b : 1 \leq a \neq b \leq n\} \cup \{L_a - L_b : 1 \leq a < b \leq n\}$$

and negative roots

$$\{-L_a - L_b : 1 \leq a \neq b \leq n\} \cup \{L_a - L_b : 1 \leq b < a \leq n\}.$$

In odd dimension, one finds positive roots

$$\{L_a + L_b : 1 \leq a \neq b \leq n\} \cup \{L_a - L_b : 1 \leq a < b \leq n\} \cup \{L_a : 1 \leq a \leq n\}$$

and negative roots

$$\{-L_a - L_b : 1 \leq a \neq b \leq n\} \cup \{L_a - L_b : 1 \leq b < a \leq n\} \cup \{-L_a : 1 \leq a \leq n\}.$$

In [3], we introduced the algebra $\mathfrak{so}(m, \mathbb{C})$ (up to an isomorphism) in the discrete Clifford analysis context. The generators of $\mathfrak{so}(m, \mathbb{C})$ were not given in terms of the root vectors and Cartan subalgebra, but rather by the generators $\{dR(e_{a,b}) : 1 \leq a \neq b \leq m\}$, satisfying the defining relations of $\mathfrak{so}(m, \mathbb{C})$:

$$[dR(e_{a,b}), dR(e_{c,d})] = \delta_{a,d} dR(e_{b,c}) + \delta_{b,c} dR(e_{a,d}) - \delta_{a,c} dR(e_{b,d}) - \delta_{b,d} dR(e_{a,c}). \quad (1)$$

In the following sections, we will re-establish the orthogonal Lie algebra in the discrete Clifford analysis setting, but now by determining the explicit expressions of the root vectors and Cartan subalgebra.

4 Decomposition of \mathcal{M}_k in irreducible representations

4.1 Even dimension $m = 2n$

Definition 1. We define the operators $H_a, X_{a,b}, Y_{a,b}$ and $Z_{a,b} \in \mathfrak{so}(m, \mathbb{C})$:

$$\begin{aligned} H_a &= i dR(e_{2a-1, 2a}), \quad 1 \leq a \leq n, \\ X_{a,b} &= \frac{1}{2} (dR(e_{2a-1, 2b-1}) + i dR(e_{2a-1, 2b}) - i dR(e_{2a, 2b-1}) + dR(e_{2a, 2b})), \\ Y_{a,b} &= \frac{1}{2} (dR(e_{2a-1, 2b-1}) - i dR(e_{2a-1, 2b}) - i dR(e_{2a, 2b-1}) - dR(e_{2a, 2b})), \\ Z_{a,b} &= \frac{1}{2} (dR(e_{2a-1, 2b-1}) + i dR(e_{2a-1, 2b}) + i dR(e_{2a, 2b-1}) - dR(e_{2a, 2b})), \quad 1 \leq a, b \leq n. \end{aligned}$$

Note that, because $dR(e_{a,b}) = -dR(e_{b,a})$, we find that $Y_{b,a} = -Y_{a,b}$ and $Z_{b,a} = -Z_{a,b}$. For $X_{a,b}$, we find that $X_{b,a} \neq X_{a,b}$ and that $X_{a,a} = H_a$, hence we will only consider couples (a, b) with $a \neq b$.

We will now show that these operators indeed show the expected commutator relations:

Lemma 1. The operators $H_c, X_{a,b}, Y_{a,b}$ and $Z_{a,b}$, $1 \leq a, b, c, d \leq n$, satisfy the commutator relations given in Lemma ??; in particular:

$$\begin{aligned} [H_c, Y_{a,b}] &= (\delta_{ca} + \delta_{cb}) Y_{a,b} = (L_a + L_b) (H_c) Y_{a,b}, \\ [H_c, X_{a,b}] &= (\delta_{ca} - \delta_{cb}) X_{a,b} = (L_a - L_b) (H_c) X_{a,b}, \\ [H_c, Z_{a,b}] &= -(\delta_{ca} + \delta_{cb}) Z_{a,b} = -(L_a + L_b) (H_c) Z_{a,b}, \\ [X_{a,b}, Y_{c,d}] &= \delta_{bc} Y_{a,d} - \delta_{bd} Y_{a,c}. \end{aligned}$$

In particular, $X_{a,b}$, $a < b$ resp. $Y_{a,b}$ are root vectors corresponding to the positive roots $L_a - L_b$, resp. $L_a + L_b$. Furthermore, $X_{a,b}$ with $a > b$ and $Z_{a,b}$ are root vectors corresponding to the negative roots $L_a - L_b$ resp. $-L_a - L_b$.

Proof. Since the commutator relations between the operators $dR(e_{a,b})$ are the same as those between the operators $\Omega_{a,b}$ of the harmonics, the proof is completely similar as the proof in [4]. \square

We already established in [?] that although \mathcal{M}_k is a representation of $\mathfrak{so}(2n, \mathbb{C})$, by means of the operators $dR(e_{a,b})$, acting on \mathcal{M}_k , this representation is not irreducible. The decomposition is done by splitting 1 into a sum of idempotents. We will now introduce the appropriate idempotents for this situation. For a function $P_k L$ to be an eigenfunction of the maximal abelian subgroup \mathfrak{h} , it must certainly hold that $L e_{2a-1}^\perp e_{2a}^\perp$ is again equal to L up to a (complex) constant. Consider, for $a = 1, \dots, n$, the Clifford elements

$$\begin{aligned} L_{2a-1}^\pm &= (e_{2a-1}^+ e_{2a-1}^- \pm i e_{2a-1}^+), & L_{2a}^\pm &= (e_{2a}^+ e_{2a}^- \pm e_{2a}^+), \\ M_{2a-1}^\pm &= (e_{2a-1}^- e_{2a-1}^+ \pm i e_{2a-1}^-), & M_{2a}^\pm &= (e_{2a}^- e_{2a}^+ \pm e_{2a}^-), \end{aligned}$$

For the rest of this article, we will need the following notations. For a factor $F_a \in \{L_a^\pm, M_a^\pm\}$, $a = 1, \dots, m$, denote

$$|F_a| = \begin{cases} 0, & F_a = L_a^+ \text{ or } M_a^-, \\ 1, & F_a = L_a^- \text{ or } M_a^+. \end{cases} \text{ and } \|F_a\| = \begin{cases} 0, & F_a = L_a^\pm, \\ 1, & F_a = M_a^\pm. \end{cases}$$

Furthermore, denote by \tilde{F}_a the idempotent

$$\tilde{F}_s = \begin{cases} L_a^\mp, & \text{if } F_a = L_a^\pm, \\ M_a^\mp, & \text{if } F_a = M_a^\pm. \end{cases}$$

Then $|\tilde{F}_s| = 1 - |F_s|$ and $\|\tilde{F}_s\| = \|F_s\|$.

Lemma 2. *The multiplication from the right on the idempotent $F_a \in \{L_a^\pm, M_a^\pm\}$ by e_a is given by*

$$\begin{aligned} F_{2a-1} e_{2a-1}^\perp &= (-1)^{|F_{2a}|+1} i F_{2a-1}, \\ F_{2a} e_{2a}^\perp &= (-1)^{|F_{2a}|+1} \tilde{F}_{2a}. \end{aligned}$$

As a result, for $1 \leq a \leq n$, we have that

$$F_{2a-1} F_{2a} e_{2a-1}^\perp e_{2a}^\perp = (-1)^{|F_{2a-1}|+|F_{2a}|+1} i F_{2a-1} F_{2a}.$$

We also find that for $1 \leq a < b \leq n$ and a general idempotent $F = \prod_{s=1}^m F_s$, with $F_s \in \{L_s^\pm, M_s^\pm\}$, we get

$$\begin{aligned} V_{2a-1, 2b-1} F e_{2a-1}^\perp e_{2b-1}^\perp &= (-1)^{|F_{2a-1}|+|F_{2b-1}|+\|F_{2a-1}\|+\|F_{2b-1}\|+1} F^{2a, 2b-1}, \\ V_{2a-1, 2b} F e_{2a-1}^\perp e_{2b}^\perp &= (-1)^{|F_{2a-1}|+|F_{2b}|+\|F_{2a-1}\|+\|F_{2b}\|} i F^{2a, 2b-1}, \\ V_{2a, 2b-1} F e_{2a}^\perp e_{2b-1}^\perp &= (-1)^{|F_{2a}|+|F_{2b-1}|+\|F_{2a}\|+\|F_{2b-1}\|} i F^{2a, 2b-1}, \\ V_{2a, 2b} F e_{2a}^\perp e_{2b}^\perp &= (-1)^{|F_{2a}|+|F_{2b}|+\|F_{2a}\|+\|F_{2b}\|} F^{2a, 2b-1}. \end{aligned}$$

where we denote, for $1 \leq s_1 < s_2 \leq m$:

$$F^{s_1, s_2} = F_1 F_2 \dots F_{s_1-1} \tilde{F}_{s_1} \tilde{F}_{s_1+1} \dots \tilde{F}_{s_2-1} \tilde{F}_{s_2} F_{s_2+1} F_{s_2+2} \dots F_{m-1} F_m.$$

Proof. Note that

$$\begin{aligned} L_{2a-1}^\pm e_{2a-1}^\perp &= (\mathbf{e}_{2a-1}^+ \mp i \mathbf{e}_{2a-1}^+ \mathbf{e}_{2a-1}^-) = \mp i L_{2a-1}^\pm, & L_{2a}^\pm e_{2a}^\perp &= (\mathbf{e}_{2a}^+ \mp \mathbf{e}_{2a}^+ \mathbf{e}_{2a}^-) = \mp L_{2a}^\mp, \\ M_{2a-1}^\pm e_{2a-1}^\perp &= (-\mathbf{e}_{2a-1}^- \pm i \mathbf{e}_{2a-1}^- \mathbf{e}_{2a-1}^+) = \pm i M_{2a-1}^\pm, & M_{2a}^\pm e_{2a}^\perp &= (-\mathbf{e}_{2a}^- \pm \mathbf{e}_{2a}^- \mathbf{e}_{2a}^+) = \pm M_{2a}^\mp. \end{aligned}$$

We may indeed summarize this as

$$F_{2a-1} e_{2a-1}^\perp = (-1)^{|F_{2a-1}|+1} i F_{2a-1}, \quad F_{2a} e_{2a}^\perp = (-1)^{|F_{2a}|+1} \tilde{F}_{2a}.$$

From this, it follows that

$$\tilde{F}_{2a-1} e_{2a-1}^\perp = (-1)^{|F_{2a-1}|} i \tilde{F}_{2a-1}, \quad \tilde{F}_{2a} e_{2a}^\perp = (-1)^{|F_{2a}|} F_{2a}.$$

Hence, for $F_a \in \{L_a^\pm, M_a^\pm\}$, we have

$$F_{2a-1} F_{2a} e_{2a-1}^\perp e_{2a}^\perp = F_{2a-1} e_{2a-1} \tilde{F}_{2a} e_{2a}^\perp = (-1)^{|F_{2a}|+|F_{2a}|+1} i F_{2a-1} F_{2a}.$$

Also important to note is that $e_a^\perp e_a L_a^\pm = L_a^\pm$ and $e_a^\perp e_a M_a^\pm = -M_a^\pm$ so for the idempotent $F = \prod_{s=1}^m F_s$, we find that

$$V_{a,b} F = -e_a^\perp e_a e_b^\perp e_b F = (-1)^{1+\|F_a\|+\|F_b\|} F.$$

We thus get, for $F = \prod_{s=1}^m F_s$, that

$$\begin{aligned} V_{2a-1,2b-1} F e_{2a-1}^\perp e_{2b-1}^\perp &= (-1)^{1+\|F_{2a-1}\|+\|F_{2b-1}\|} F_1 F_2 \dots F_m e_{2a-1}^\perp e_{2b-1}^\perp \\ &= (-1)^{1+\|F_{2a-1}\|+\|F_{2b-1}\|} F_1 \dots F_{2a-2} F_{2a-1} e_{2a-1}^\perp \tilde{F}_{2a} \dots \tilde{F}_{2b-1} e_{2b-1}^\perp F_{2b} F_{2b+1} \dots F_m \\ &= (-1)^{|F_{2a-1}|+|F_{2b-1}|+\|F_{2a-1}\|+\|F_{2b-1}\|} i^2 F_1 F_2 \dots F_{2a-2} F_{2a-1} \tilde{F}_{2a} \dots \tilde{F}_{2b-1} F_{2b} F_{2b+1} \dots F_m \\ &= (-1)^{|F_{2a-1}|+|F_{2b-1}|+\|F_{2a-1}\|+\|F_{2b-1}\|+1} F^{2a,2b-1}. \end{aligned}$$

Analogously, we find that

$$\begin{aligned} V_{2a-1,2b} F e_{2a-1}^\perp e_{2b}^\perp &= (-1)^{1+\|F_{2a-1}\|+\|F_{2b}\|} F_1 F_2 \dots F_m e_{2a-1}^\perp e_{2b}^\perp \\ &= (-1)^{1+\|F_{2a-1}\|+\|F_{2b}\|} F_1 \dots F_{2a-2} F_{2a-1} e_{2a-1}^\perp \tilde{F}_{2a} \dots \tilde{F}_{2b} e_{2b}^\perp F_{2b+1} F_{2b+2} \dots F_m \\ &= (-1)^{|F_{2a-1}|+|F_{2b}|+\|F_{2a-1}\|+\|F_{2b}\|} i F_1 F_2 \dots F_{2a-2} F_{2a-1} \tilde{F}_{2a} \dots \tilde{F}_{2b-1} F_{2b} F_{2b+1} \dots F_m \\ &= (-1)^{|F_{2a-1}|+|F_{2b}|+\|F_{2a-1}\|+\|F_{2b}\|} i F^{2a,2b-1}. \end{aligned}$$

Also

$$\begin{aligned} V_{2a,2b-1} F e_{2a}^\perp e_{2b-1}^\perp &= (-1)^{1+\|F_{2a}\|+\|F_{2b-1}\|} F_1 F_2 \dots F_m e_{2a}^\perp e_{2b-1}^\perp \\ &= (-1)^{1+\|F_{2a}\|+\|F_{2b-1}\|} F_1 \dots F_{2a-1} F_{2a} e_{2a}^\perp \tilde{F}_{2a+1} \dots \tilde{F}_{2b-1} e_{2b-1}^\perp F_{2b} F_{2b+1} \dots F_m \\ &= (-1)^{|F_{2a}|+|F_{2b-1}|+\|F_{2a}\|+\|F_{2b-1}\|} i F_1 F_2 \dots F_{2a-1} \tilde{F}_{2a} \tilde{F}_{2a+1} \dots \tilde{F}_{2b-1} F_{2b} F_{2b+1} \dots F_m \\ &= (-1)^{|F_{2a}|+|F_{2b-1}|+\|F_{2a}\|+\|F_{2b-1}\|} i F^{2a,2b-1}. \end{aligned}$$

Finally

$$\begin{aligned} V_{2a,2b} F e_{2a}^\perp e_{2b}^\perp &= (-1)^{1+\|F_{2a}\|+\|F_{2b}\|} F_1 F_2 \dots F_m e_{2a}^\perp e_{2b}^\perp \\ &= (-1)^{1+\|F_{2a}\|+\|F_{2b}\|} F_1 \dots F_{2a-1} F_{2a} e_{2a}^\perp \tilde{F}_{2a+1} \dots \tilde{F}_{2b} e_{2b}^\perp F_{2b+1} \dots F_m \\ &= (-1)^{|F_{2a}|+|F_{2b}|+\|F_{2a}\|+\|F_{2b}\|} F_1 F_2 \dots F_{2a-1} \tilde{F}_{2a} \tilde{F}_{2a+1} \dots \tilde{F}_{2b-1} F_{2b} F_{2b+1} \dots F_m \\ &= (-1)^{|F_{2a}|+|F_{2b}|+\|F_{2a}\|+\|F_{2b}\|} F^{2a,2b-1}. \end{aligned}$$

□

Consider the basic monogenic functions

$$g_{2k} = ((\xi_2 - \xi_1)(\xi_2 + \xi_1))^k, \quad g_{2k+1} = (\xi_2 - \xi_1)((\xi_2 + \xi_1)(\xi_2 - \xi_1))^k.$$

From now on we denote $(k)'_+ = (k + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ and $(k)'_- = (k + \frac{1}{2}, \frac{1}{2}, \dots, -\frac{1}{2})$. We will show under which conditions on the idempotent $F = \prod_{s=1}^n F_s$, the space $\text{span}_{\mathbb{C}} \{g_k F\}$ is a weight space of \mathfrak{h} with weight $(k)'_+$ resp. $(k)'_-$.

Lemma 3. *The polynomial $g_k F \in \mathcal{M}_k$, $F = \prod_{s=1}^m F_s$ with $F_s \in \{L_s^\pm, M_s^\pm\}$, is a weight vector of $\mathfrak{so}(m, \mathbb{C})$ with*

- *weight $(k)'_+$ when $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ is even and $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ is even for $2 \leq a \leq n$.*
- *weight $(k)'_-$ when $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ is even, $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ is even for $2 \leq a \leq n-1$ and $\|F_{2n-1}\| + \|F_{2n}\| + |F_{2n-1}| + |F_{2n}|$ is odd.*

Proof. We consider the action of the Cartan subalgebra-elements H_s , $1 \leq s \leq n$, on the $g_k F$. Since g_k only contains ξ_1 and ξ_2 , we will first consider H_1 :

$$H_1(g_k F) = i V_{12} \left(L_{12} - \frac{1}{2} \right) g_k F e_1^\perp e_2^\perp.$$

We will also denote

$$f_{2k} = ((\xi_2 + \xi_1)(\xi_2 - \xi_1))^k, \quad f_{2k+1} = (\xi_2 + \xi_1)((\xi_2 - \xi_1)(\xi_2 + \xi_1))^k.$$

In [] it was established that $\partial_j g_k = (-1)^j k f_{k-1}$:

$$L_{12} g_k = (\xi_1 \partial_2 + \xi_2 \partial_1) g_k = k(\xi_1 - \xi_2) f_{k-1} = -k(\xi_2 - \xi_1) f_{k-1} = -k g_k.$$

We thus get that

$$H_1(g_k F) = i \left(-k - \frac{1}{2} \right) V_{12} g_k F e_1^\perp e_2^\perp.$$

Now we will show that $V_{12} g_k = (-1)^k g_k V_{12}$: consider again $V_{12} = -e_1^\perp e_1 e_2^\perp e_2$, then

$$\begin{aligned} e_1^\perp e_1 \xi_1 &= (\mathbf{e}_1^+ \mathbf{e}_1^- - \mathbf{e}_1^- \mathbf{e}_1^+) (X_1^+ \mathbf{e}_1^- + X_1^- \mathbf{e}_1^+) = (-X_1^+ \mathbf{e}_1^- + X_1^- \mathbf{e}_1^+) \\ &= (X_1^+ \mathbf{e}_1^- + X_1^- \mathbf{e}_1^+) (-\mathbf{e}_1^+ \mathbf{e}_1^- + \mathbf{e}_1^- \mathbf{e}_1^+) = -\xi_1 e_1^\perp e_1, \\ e_1^\perp e_1 \xi_2 &= \xi_2 e_1^\perp e_1. \end{aligned}$$

Hence

$$\begin{aligned} V_{12}(\xi_2 \pm \xi_1) &= -e_1^\perp e_1 e_2^\perp e_2 (\xi_2 \pm \xi_1) = (\xi_2 \pm \xi_1) e_1^\perp e_1 e_2^\perp e_2 = -(\xi_2 \pm \xi_1) V_{12}, \\ V_{12}((\xi_2 - \xi_1)(\xi_2 + \xi_1)) &= ((\xi_2 - \xi_1)(\xi_2 + \xi_1)) V_{12} \end{aligned}$$

and so $V_{12} g_k = (-1)^k g_k V_{12}$. Applying this, we find that

$$\begin{aligned} H_1 (g_k F) &= (-1)^{k+1} i \left(k + \frac{1}{2} \right) g_k V_{12} F e_1^\perp e_2^\perp \\ &= (-1)^{k+|F_1|+|F_2|+\|F_1\|+\|F_2\|} \left(k + \frac{1}{2} \right) g_k F. \end{aligned}$$

To be a weight vector with weight $k + \frac{1}{2}$, it must hold that

$$k + |F_1| + |F_2| + \|F_1\| + \|F_2\| \text{ is even.}$$

We thus find 8 possible combinations for $F_1 F_2$:

- k even:

$$F_1 F_2 \in \{L_1^+ L_2^+, L_1^- L_2^-, L_1^+ M_2^+, L_1^- M_2^-, M_1^+ L_2^+, M_1^- M_2^+, M_1^- L_2^-, M_1^+ M_2^-\}.$$

- k odd:

$$F_1 F_2 \in \{L_1^+ L_2^-, L_1^- L_2^+, L_1^+ M_2^-, L_1^- M_2^+, M_1^+ L_2^-, M_1^- M_2^-, M_1^- L_2^+, M_1^+ M_2^+\}.$$

Next, we consider H_a , $1 < a \leq n$. Since the generator g_k only contains ξ_1 and ξ_2 , it vanishes under the action of $L_{2a-1,2a}$. Note that $V_{2a-1,2a} g_k = g_k V_{2a-1,2a}$ since g_k contains only \mathbf{e}_1^\pm and \mathbf{e}_2^\pm . Thus

$$\begin{aligned} H_a (g_k F) &= -\frac{i}{2} V_{2a-1,2a} g_k F e_{2a-1}^\perp e_{2a}^\perp = -\frac{i}{2} g_k V_{2a-1,2a} F e_{2a-1}^\perp e_{2a}^\perp \\ &= (-1)^{\|F_{2a-1}\|+\|F_{2a}\|+|F_{2a-1}|+|F_{2a}|+1} i \frac{i}{2} g_k F \\ &= (-1)^{\|F_{2a-1}\|+\|F_{2a}\|+|F_{2a-1}|+|F_{2a}|} \frac{1}{2} g_k F. \end{aligned}$$

This equals $+\frac{1}{2} g_k F$ when $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ is even and $-\frac{1}{2} g_k F$ when $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ is odd. We may thus conclude that the statement holds. We find that $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ is even for $F_{2a-1} F_{2a}$ in

$$\{L_{2a-1}^+ L_{2a}^+, L_{2a-1}^- L_{2a}^-, L_{2a-1}^+ M_{2a}^+, L_{2a-1}^- M_{2a}^-, M_{2a-1}^+ L_{2a}^+, M_{2a-1}^- M_{2a}^+, M_{2a-1}^- L_{2a}^-, M_{2a-1}^+ M_{2a}^-\}$$

and odd for $F_{2a-1} F_{2a}$ in

$$\{L_{2a-1}^+ L_{2a}^-, L_{2a-1}^- L_{2a}^+, L_{2a-1}^+ M_{2a}^-, L_{2a-1}^- M_{2a}^+, M_{2a-1}^+ L_{2a}^-, M_{2a-1}^- M_{2a}^-, M_{2a-1}^- L_{2a}^+, M_{2a-1}^+ M_{2a}^+\}.$$

□

Remark 1. In particular, we find that $g_{2k} \prod_{s=1}^m L_s^+$ respectively $g_{2k+1} L_1^+ L_2^- \prod_{s=3}^m L_s^+$ are weight vectors of \mathfrak{h} in \mathcal{M}_{2k} resp. \mathcal{M}_{2k+1} of weight $(2k)'_+$ resp. $(2k+1)'_+$.

Corollary 1. There are 2^{2m-n} weight vectors $g_k F$, with F one of the above mentioned idempotents, of weight $(k)'_+$ and 2^{2m-n} weight vectors $g_k F$, with F one of the above mentioned idempotents, with weight $(k)'_-$.

Proof. To obtain weight $(k)'_+$, one has eight choices for each factor $F_{2s-1} F_{2s}$ in F , $1 \leq s \leq n$. We thus get $8^n = 2^{3n} = 2^{4n-n} = 2^{2m-n}$ choices for the idempotent F . The same count holds for the weight $(k)'_-$. \square

We will now show that the weight vectors, defined in Lemma 3 are actually highest weight vectors, i.e. that they vanish under the action of all positive roots.

Lemma 4. The polynomials $g_k F$, with

- $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ even
- $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ even, $\forall 2 \leq a \leq n-1$, and
- $\|F_{2n-1}\| + \|F_{2n}\| + |F_{2n-1}| + |F_{2n}|$ even resp. odd

are highest weight spaces with highest weight $(k)'_+$ resp. $(k)'_-$, i.e.

$$H_a(g_k F) = \left(\delta_{1a} \left(k + \frac{1}{2} \right) + \frac{1}{2} \sum_{j \neq 1} \delta_{ja} \right) g_k F$$

and

$$X_{a,b}(g_k F) = 0, \forall (a,b), a < b, \quad Y_{a,b}(g_k F) = 0, \forall (a,b), a \neq b.$$

Proof. We have already shown that these $g_k F$ are weight vectors with weight $(k)'_{\pm}$. We now show that $g_k F$ vanishes under the action of $X_{a,b}$, $a < b$, and $Y_{a,b}$, $a \neq b$. Note that $X_{a,b}(g_k F)$ denotes the action of the operator $X_{a,b}$ on $g_k F$; this is not a multiplication.

We first consider the action of $X_{a,b}$ on $g_k F$. We make a distinction between $a = 1$ and $a \neq 1$. First let $1 = a < b$, then

$$\begin{aligned} 2X_{1,b} g_k F &= (dR(e_{1,2b-1}) + i dR(e_{1,2b}) - i dR(e_{2,2b-1}) + dR(e_{2,2b})) g_k F \\ &= V_{1,2b-1} \left(\xi_{2b-1} \partial_1 - \frac{1}{2} \right) g_k F e_1^\perp e_{2b-1}^\perp + i V_{1,2b} \left(\xi_{2b} \partial_1 - \frac{1}{2} \right) g_k F e_1^\perp e_{2b}^\perp \\ &\quad - i V_{2,2b-1} \left(\xi_{2b-1} \partial_2 - \frac{1}{2} \right) g_k F e_2^\perp e_{2b-1}^\perp + V_{2,2b} \left(\xi_{2b} \partial_2 - \frac{1}{2} \right) g_k F e_2^\perp e_{2b}^\perp \\ &= V_{1,2b-1} \left(-k \xi_{2b-1} f_{k-1} - \frac{1}{2} g_k \right) F e_1^\perp e_{2b-1}^\perp + i V_{1,2b} \left(-k \xi_{2b} f_{k-1} - \frac{1}{2} g_k \right) F e_1^\perp e_{2b}^\perp \\ &\quad - i V_{2,2b-1} \left(k \xi_{2b-1} f_{k-1} - \frac{1}{2} g_k \right) F e_2^\perp e_{2b-1}^\perp + V_{2,2b} \left(k \xi_{2b} f_{k-1} - \frac{1}{2} g_k \right) F e_2^\perp e_{2b}^\perp. \end{aligned}$$

Now we use

$$V_{1,2b-1} \xi_{2b-1} = -\xi_{2b-1} V_{1,2b-1}, \quad V_{1,2b} \xi_{2b} = -\xi_{2b} V_{1,2b}.$$

Furthermore, since for $b \neq 1, 2$,

$$\begin{aligned} V_{1,b} (\xi_2 \pm \xi_1) &= (\xi_2 \mp \xi_1) V_{1,b}, \\ V_{2,b} (\xi_2 \pm \xi_1) &= (-\xi_2 \pm \xi_1) V_{2,b} = -(\xi_2 \mp \xi_1) V_{2,b}, \end{aligned}$$

we find, for $j \in \{2b-1, 2b\}$:

$$\begin{aligned} V_{1,j} g_k &= V_{1,j} (\xi_2 - \xi_1) (\xi_2 + \xi_1) (\xi_2 - \xi_1) \dots = (\xi_2 + \xi_1) (\xi_2 - \xi_1) (\xi_2 + \xi_1) \dots V_{1,j} = f_k V_{1,j}, \\ V_{1,j} f_k &= V_{1,j} (\xi_2 + \xi_1) (\xi_2 - \xi_1) (\xi_2 + \xi_1) \dots = (\xi_2 - \xi_1) (\xi_2 + \xi_1) (\xi_2 - \xi_1) \dots V_{1,j} = g_k V_{1,j}, \\ V_{2,j} g_k &= V_{2,j} (\xi_2 - \xi_1) (\xi_2 + \xi_1) (\xi_2 - \xi_1) \dots = (-1)^k f_k V_{2,j}, \\ V_{2,j} f_k &= V_{2,j} (\xi_2 + \xi_1) (\xi_2 - \xi_1) (\xi_2 + \xi_1) \dots = (-1)^k g_k V_{2,j}. \end{aligned}$$

We get that

$$\begin{aligned} 2 X_{1,b} g_k F &= \left(k \xi_{2b-1} g_{k-1} - \frac{1}{2} f_k \right) V_{1,2b-1} F e_1^\perp e_{2b-1}^\perp + i \left(k \xi_{2b} g_{k-1} - \frac{1}{2} f_k \right) V_{1,2b} F e_1^\perp e_{2b}^\perp \\ &\quad - i \left((-1)^{1+k-1} k \xi_{2b-1} g_{k-1} - (-1)^k \frac{1}{2} f_k \right) V_{2,2b-1} F e_2^\perp e_{2b-1}^\perp \\ &\quad + \left((-1)^{1+k-1} k \xi_{2b} g_{k-1} - \frac{1}{2} (-1)^k f_k \right) V_{2,2b} F e_2^\perp e_{2b}^\perp. \end{aligned}$$

We now use that

$$\begin{aligned} V_{1,2b-1} F e_1^\perp e_{2b-1}^\perp &= (-1)^{|F_1|+|F_{2b-1}|+\|F_1\|+\|F_{2b-1}\|+1} F^{2,2b-1}, \\ V_{1,2b} F e_1^\perp e_{2b}^\perp &= (-1)^{|F_1|+|F_{2b}|+\|F_1\|+\|F_{2b}\|} i F^{2,2b-1}, \\ V_{2,2b-1} F e_2^\perp e_{2b-1}^\perp &= (-1)^{|F_2|+|F_{2b-1}|+\|F_2\|+\|F_{2b-1}\|} i F^{2,2b-1}, \\ V_{2,2b} F e_2^\perp e_{2b}^\perp &= (-1)^{|F_2|+|F_{2b}|+\|F_2\|+\|F_{2b}\|} F^{2,2b-1}. \end{aligned}$$

This results in

$$\begin{aligned} 2 X_{1,b} g_k F &= (-1)^{|F_1|+\|F_1\|+|F_{2b-1}|+\|F_{2b-1}\|} \\ &\quad \left(-k \xi_{2b-1} g_{k-1} + \frac{1}{2} f_k + (-1)^{|F_{2b-1}|+\|F_{2b-1}\|+|F_{2b}|+\|F_{2b}\|} \left(-k \xi_{2b} g_{k-1} + \frac{1}{2} f_k \right) \right. \\ &\quad + \left((-1)^k k \xi_{2b-1} g_{k-1} - (-1)^k \frac{1}{2} f_k \right) (-1)^{|F_1|+\|F_1\|+|F_2|+\|F_2\|} \\ &\quad \left. + \left((-1)^k k \xi_{2b} g_{k-1} - \frac{1}{2} (-1)^k f_k \right) (-1)^{|F_1|+\|F_1\|+|F_2|+\|F_2\|+|F_{2b-1}|+|F_{2b}|+\|F_{2b-1}\|+\|F_{2b}\|} \right) F^{2,2b-1}. \end{aligned}$$

We thus see that this vanishes when

$$k + |F_1| + \|F_1\| + |F_2| + \|F_2\|$$

is even.

For $1 < a < b \leq n$ we get that

$$\begin{aligned} 2 X_{a,b} g_k F &= -\frac{1}{2} \left(V_{2a-1,2b-1} g_k F e_{2a-1}^\perp e_{2b-1}^\perp + i V_{2a-1,2b} g_k F e_{2a-1}^\perp e_{2b}^\perp \right. \\ &\quad \left. - i V_{2a,2b-1} g_k F e_{2a}^\perp e_{2b-1}^\perp + V_{2a,2b} g_k F e_{2a}^\perp e_{2b}^\perp \right) \\ &= -\frac{1}{2} g_k \left((-1)^{|F_{2a-1}|+|F_{2b-1}|+\|F_{2a-1}\|+\|F_{2b-1}\|+1} + (-1)^{|F_{2a-1}|+|F_{2b}|+\|F_{2a-1}\|+\|F_{2b}\|+1} \right. \\ &\quad \left. + (-1)^{|F_{2a}|+|F_{2b-1}|+\|F_{2a}\|+\|F_{2b-1}\|} + (-1)^{|F_{2a}|+|F_{2b}|+\|F_{2a}\|+\|F_{2b}\|} \right) F^{2a,2b-1} \\ &= -\frac{1}{2} g_k (-1)^{|F_{2a-1}|+\|F_{2a-1}\|+|F_{2b-1}|+\|F_{2b-1}\|} \\ &\quad \left(-1 + (-1)^{|F_{2b-1}|+\|F_{2b-1}\|+|F_{2b}|+\|F_{2b}\|+1} \right. \\ &\quad \left. + (-1)^{|F_{2a-1}|+\|F_{2a-1}\|+|F_{2a}|+\|F_{2a}\|} + (-1)^{|F_{2a-1}|+\|F_{2a-1}\|+|F_{2a}|+\|F_{2a}\|+|F_{2a}|+\|F_{2a}\|+|F_{2b}|+\|F_{2b}\|} \right) F^{2a,2b-1}. \end{aligned}$$

This will be zero when $|F_{2a-1}|+\|F_{2a-1}\|+|F_{2a}|+\|F_{2a}\|$ is even, and this for all $2 \leq a \leq n-1$.

Note that:

$$\begin{aligned} X_{a,b} &= \frac{1}{2} (dR(e_{2a-1,2b-1}) + i dR(e_{2a-1,2b}) - i dR(e_{2a,2b-1}) + dR(e_{2a,2b})), \\ Y_{a,b} &= \frac{1}{2} (dR(e_{2a-1,2b-1}) - i dR(e_{2a-1,2b}) - i dR(e_{2a,2b-1}) - dR(e_{2a,2b})). \end{aligned}$$

If we apply the appropriate change of sign in the second and last term of previous calculations, we immediately get that $Y_{a,b}(g_k F) = 0$ for $a < b$. Since $Y_{a,b} = -Y_{b,a}$, this will also be zero for $a > b$. \square

Remark 2. In particular, the polynomial $g_{2k} \prod_{s=1}^m L_s^+$ and $g_{2k+1} L_1^+ L_2^- \prod_{s=3}^m L_s^+$ are highest weight vectors with weight $(2k)_+'$ resp. $(2k+1)_+'$.

Remark 3. The dimension of $(k)_\pm'$ is (see [6])

$$2^{n-1} \binom{k+m-2}{k}.$$

As the dimension of \mathcal{M}_k equals

$$2^{2m} \binom{k+m-2}{k} = 2^{4n} \binom{k+m-2}{k}$$

and as we found 2^{3n} isomorphic copies of $(k)_+'$ combined with 2^{3n} copies of $(k)_-'$, the space \mathcal{M}_k is fully decomposed in 2^{3n} copies of $(k)_+'$ and 2^{3n} copies of $(k)_-'$.

Definition 2. We define the positive resp. negative spinorspace \mathbb{S}_{2n}^\pm as the image under $\mathfrak{so}(m, \mathbb{C})$ of the idempotents $\prod_{s=1}^m L_s^+$, resp. $\left(\prod_{s=1}^{m-1} L_s^+\right) L_m^-$:

$$\mathbb{S}_{2n}^+ = \mathfrak{so}(m, \mathbb{C}) \left(\text{span}_{\mathbb{C}} \{ L_1^+ L_2^+ \dots L_{2n-1}^+ L_{2n}^+ \} \right)$$

and

$$\mathbb{S}_{2n}^- = \mathfrak{so}(m, \mathbb{C}) \left(\text{span}_{\mathbb{C}} \{ L_1^+ L_2^+ \dots L_{2n-1}^+ L_{2n}^- \} \right).$$

The elements $L_1^+ L_2^+ \dots L_{2n-1}^+ L_{2n}^+$, resp. $L_1^+ L_2^+ \dots L_{2n-1}^+ L_{2n}^-$ are highest weight vectors with weight $(0)'_+ = (\frac{1}{2}, \dots, \frac{1}{2})$ resp. $(0)'_- = (\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$ and they thus generate irreducible representations with the same weight.

Example 1. Let $m = 4$ (i.e. $n = 2$) and consider $L = L_1^+ L_2^+ L_3^+ L_4^+$. The Lie algebra $\mathfrak{so}(4, \mathbb{C})$ is given in this context by

$$\text{span}_{\mathbb{C}} \{ dR(e_{12}), dR(e_{13}), dR(e_{14}), dR(e_{23}), dR(e_{24}), dR(e_{34}) \}.$$

The elements $dR(e_{12})$ and $dR(e_{34})$ return L up to complex constant. The other four rotations give us (up to a complex constant) the idempotent $L_1^+ L_2^- L_3^- L_4^+$. Hence

$$\mathbb{S}_4^+ = \text{span}_{\mathbb{C}} \{ L_1^+ L_2^+ L_3^+ L_4^+, L_1^+ L_2^- L_3^- L_4^+ \}.$$

Starting from $L_1^+ L_2^+ L_3^+ L_4^-$, the rotations $dR(e_{13})$, $dR(e_{14})$, $dR(e_{23})$ and $dR(e_{24})$ lead us to the idempotent $L_1^+ L_2^- L_3^- L_4^-$ which shows that

$$\mathbb{S}_4^- = \text{span}_{\mathbb{C}} \{ L_1^+ L_2^+ L_3^+ L_4^-, L_1^+ L_2^- L_3^- L_4^- \}.$$

The (positive/negative) spinorspace \mathbb{S}_{2n}^\pm is 2^{n-1} -dimensional.

In general, the elements $dR(e_{2a-1, 2a})$ acting on an idempotent return the same idempotent up to a multiplicative complex factor. Since, for $1 \leq a < b \leq n$:

$$\begin{aligned} V_{2a-1, 2b-1} L e_{2a-1}^\perp e_{2b-1}^\perp &= -L^{a,b}, & V_{2a-1, 2b} L e_{2a-1}^\perp e_{2b}^\perp &= i L^{a,b}, \\ V_{2a, 2b-1} L e_{2a}^\perp e_{2b-1}^\perp &= i L^{a,b}, & V_{2a, 2b} L e_{2a}^\perp e_{2b}^\perp &= L^{a,b}. \end{aligned}$$

with $L^{a,b} = L_1^+ L_2^+ \dots L_{2a-1}^+ L_{2a}^- \dots L_{2b-1}^- L_{2b}^+ \dots L_{2n-1}^+ L_{2n}^+$, we see that $dR(e_{a,b})$ acting on

$$L = L_1^+ L_2^+ \dots L_{2n-1}^+ L_{2n}^+$$

changes the sign of an even number of L_a 's. The operator always leaves L_1^+ and L_{2n}^+ invariant. The resulting idempotent will always have an even number of minus-signs. Starting from the idempotent L with all plus-signs, we thus get all possible idempotents of the following form:

$$L_1^+ \underbrace{\quad} \underbrace{\quad} \dots \underbrace{\quad} L_{2n}^+.$$

where each place $\underbrace{\quad}$ consists of either $L_{2a}^+ L_{2a+1}^+$ or $L_{2a}^- L_{2a+1}^-$, $1 \leq a \leq n-1$. We get 2^{n-1} spinors belonging to the positive spinorspace and we have the following weight space decomposition

$$\mathbb{S}_{2n}^+ = \bigoplus V_{(\pm\frac{1}{2}, \pm\frac{1}{2}, \dots, \pm\frac{1}{2})},$$

where the sum goes over all weights with an even number of minus-signs. The highest weight remains $(\frac{1}{2}, \dots, \frac{1}{2})$ and the highest weight vector is L .

Starting from $L_1^+ L_2^+ \dots L_{2n-1}^+ L_{2n}^-$, we will generate all possible idempotents of the following form:

$$L_1^+ \underbrace{\quad} \underbrace{\quad} \dots \underbrace{\quad} L_{2n}^-.$$

where each place $\underbrace{\quad}$ consists of either $L_{2a}^+ L_{2a+1}^+$ or $L_{2a}^- L_{2a+1}^-$, $1 \leq a \leq n-1$. We thus also get 2^{n-1} spinors belonging to the negative spinorspace and the following weight space decomposition:

$$\mathbb{S}_{2n}^- = \bigoplus V_{(\pm\frac{1}{2}, \pm\frac{1}{2}, \dots, \pm\frac{1}{2})},$$

where the sum goes over all weights with an odd number of minus-signs. The highest weight is still $(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$ and the highest weight vector is $L_1^+ L_2^+ \dots L_{2n-1}^+ L_{2n}^-$.

4.2 Odd dimension $m = 2n + 1$

We now extend the set of generators H_a , $X_{a,b}$, $Y_{a,b}$ and $Z_{a,b}$ of $\mathfrak{so}(m, \mathbb{C})$ with $2n$ mappings

$$\begin{aligned} U_a &= \frac{1}{\sqrt{2}} (dR(e_{2a-1,m}) - i dR(e_{2a,m})), \\ V_a &= \frac{1}{\sqrt{2}} (dR(e_{2a-1,m}) + i dR(e_{2a,m})), \end{aligned}$$

where $1 \leq a \leq n$. With the addition of these $2n$ mappings, we are again able to reconstruct all original $dR(e_{a,b})$'s since $\sqrt{2} dR(e_{2a-1,m}) = U_a + V_a$ and $-\sqrt{2} i dR(e_{2a,m}) = U_a - V_a$.

The classic commutator relations follow immediately.

Lemma 5. *For $1 \leq a, b \leq n$, it holds that*

$$\begin{aligned} [H_a, U_b] &= \delta_{ab} U_b = L_b(H_a) U_b, \\ [H_a, V_b] &= -\delta_{ab} V_b = -L_b(H_a) V_b. \end{aligned}$$

In particular, U_b is a root vector corresponding to the positive root L_b and V_b is a root vector corresponding with the negative root $-L_b$, $\forall 1 \leq b \leq n$.

Lemma 6. *The operators U_c and V_d satisfy the following additional commutator relations with $X_{a,b}$, $Y_{a,b}$ and $Z_{a,b}$, $1 \leq a, b, c, d \leq n$:*

$$\begin{aligned} [U_c, X_{a,b}] &= -\delta_{cb} U_a, & [V_c, X_{a,b}] &= \delta_{ca} V_b, \\ [U_c, Y_{a,b}] &= 0, & [V_c, Y_{a,b}] &= \delta_{ca} U_b - \delta_{cb} U_a, \\ [U_c, Z_{a,b}] &= -\delta_{cb} V_a + \delta_{ca} V_b, & [V_c, Z_{a,b}] &= 0, \\ [U_c, U_d] &= -Y_{c,d}, \quad c \neq d, & [V_c, V_d] &= -Z_{c,d}, \quad c \neq d, \\ [U_c, V_d] &= \begin{cases} -X_{c,d}, & c \neq d, \\ -H_c, & c = d. \end{cases} \end{aligned}$$

Proof. The statements follow immediately from the definitions of U_c and V_d and from the defining relations (1) which the operators $dR(e_{a,b})$ satisfy. \square

We now introduce four extra idempotents

$$L_m^\pm = (\mathbf{e}_m^+ \mathbf{e}_m^- \pm i \mathbf{e}_m^+), \quad M_m^\pm = (\mathbf{e}_m^- \mathbf{e}_m^+ \pm \mathbf{e}_m^-)$$

and denote

$$L = \prod_{a=1}^n (L_{2a-1}^+ L_{2a}^+) L_m^+, \quad L' = L_1^+ L_2^- \prod_{a=2}^n (L_{2a-1}^+ L_{2a}^+) L_m^+.$$

We will now show that the highest weight vectors of weight $(k)'_+$ from the even-dimensional setting are still highest weight vectors with weight $(k)'_+$ when we add one of the four possible extra factors to the idempotent F .

Lemma 7. *The weight vectors $g_k F$, $F = \prod_{s=1}^m F_s$ with $F_s \in \{L_s^\pm, M_s^\pm\}$, such that*

- $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ even
- $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ even, $\forall 2 \leq a \leq n$,

vanish under the operator U_a , $1 \leq a \leq n$, i.e.

$$U_a (g_k F) = 0, \quad \forall 1 \leq a \leq n.$$

Proof. Consider

$$\sqrt{2} U_a (g_k F) = (dR(e_{2a-1,m}) - i dR(e_{2a,m})) g_k F.$$

Since g_k contains only ξ_1 and ξ_2 , we will make a distinction between $a = 1$ and $a \neq 1$. We start with assuming that $a = 1$. Then

$$\begin{aligned} \sqrt{2} U_1 (g_k F) &= i V_{1,m} \left(\xi_m \partial_1 - \frac{1}{2} \right) g_k F e_1^\perp e_m^\perp - i^2 V_{2,m} \left(\xi_m \partial_2 - \frac{1}{2} \right) g_k F e_2^\perp e_m^\perp \\ &= i V_{1,m} \left(-k \xi_m f_{k-1} - \frac{1}{2} g_k \right) F e_1^\perp e_m^\perp + V_{2,m} \left(k \xi_m f_{k-1} - \frac{1}{2} g_k \right) F e_2^\perp e_m^\perp. \end{aligned}$$

Now we again use that, for $j \neq 1, 2$:

$$V_{1,j} f_k = g_k V_{1,j}, \quad V_{2,j} f_k = (-1)^k g_k V_{2,j}, \quad V_{1,j} g_k = f_k V_{1,j}, \quad V_{2,j} g_k = (-1)^k f_k V_{2,j}.$$

Hence

$$\begin{aligned} & \sqrt{2} U_1 (g_k F) \\ &= i \left(k \xi_m V_{1,m} f_{k-1} - \frac{1}{2} V_{1,m} g_k \right) F e_1^\perp e_m^\perp + \left(-k \xi_m V_{2,m} f_{k-1} - \frac{1}{2} V_{2,m} g_k \right) F e_2^\perp e_m^\perp \\ &= i \left(k \xi_m g_{k-1} - \frac{1}{2} f_k \right) V_{1,m} F e_1^\perp e_m^\perp + \left((-1)^{k-1+1} k \xi_m g_{k-1} - \frac{1}{2} (-1)^k f_k \right) V_{2,m} F e_2^\perp e_m^\perp. \end{aligned}$$

We complete the proof by noting that

$$\begin{aligned} V_{1,m} F e_1^\perp e_m^\perp &= (-1)^{\|F_1\|+\|F_m\|+1} F e_1^\perp e_m^\perp = (-1)^{\|F_1\|+\|F_m\|+1} F_1 e_1^\perp \tilde{F}_2 \dots \tilde{F}_{m-1} \tilde{F}_m e_m^\perp \\ &= (-1)^{\|F_1\|+\|F_1\|+\|F_m\|+\|F_m\|} i^2 F_1 \tilde{F}_2 \dots \tilde{F}_{m-1} \tilde{F}_m \\ &= (-1)^{\|F_1\|+\|F_1\|+\|F_m\|+\|F_m\|+1} F^{2,m}, \\ V_{2,m} F e_2^\perp e_m^\perp &= (-1)^{\|F_2\|+\|F_m\|+1} F_1 F_2 e_2^\perp \tilde{F}_3 \tilde{F}_4 \dots \tilde{F}_{m-1} \tilde{F}_m e_m^\perp \\ &= (-1)^{\|F_2\|+\|F_2\|+\|F_m\|+\|F_m\|} i F^{2,m}. \end{aligned}$$

Thus $\sqrt{2} U_1 (g_k F)$ will be zero since $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ is even.

When $a \neq 1$, the action of $L_{2a-1,m}$ on $g_k F$ results in zero hence

$$\begin{aligned} \sqrt{2} U_a (g_k F) &= -\frac{i}{2} V_{2a-1,m} g_k F e_{2a-1}^\perp e_m^\perp - \frac{1}{2} V_{2a,m} g_k F e_{2a}^\perp e_m^\perp \\ &= -\frac{1}{2} g_k \left(i V_{2a-1,m} F e_{2a-1}^\perp e_m^\perp + V_{2a,m} F e_{2a}^\perp e_m^\perp \right) \\ &= -\frac{1}{2} g_k \left((-1)^{\|F_{2a-1}\|+\|F_{2a-1}\|+\|F_m\|+\|F_m\|} i + (-1)^{\|F_{2a}\|+\|F_{2a}\|+\|F_m\|+\|F_m\|+1} i \right) F^{2a,m} \\ &= -\frac{i}{2} g_k (-1)^{\|F_{2a-1}\|+\|F_{2a-1}\|+\|F_m\|+\|F_m\|} \left(1 + (-1)^{\|F_{2a-1}\|+\|F_{2a-1}\|+\|F_{2a}\|+\|F_{2a}\|+1} \right) F^{2a,m}. \end{aligned}$$

This will be zero when $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ is even, $\forall 2 \leq a \leq n$. Note that the highest weight vectors of weight $(k)'_-$ will not vanish under the action of U_n . \square

Corollary 2. *The polynomials $g_k F$ with $F = \prod_{s=1}^m F_s$, $F_s \in \{L_s^\pm, M_s^\pm\}$ such that*

- $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ even
- $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ even, $\forall 2 \leq a \leq n$,

are highest weight vectors, in \mathcal{M}_k of weight $(k)'_+$. In particular, $g_{2k} \prod_{s=1}^m L_s^+$ and $g_{2k+1} L_1^+ L_2^- \prod_{s=3}^m L_s^+$ are highest weight vectors of weight $(2k)'_+$ resp. $(2k+1)'_+$.

Note that the choice for the last factor $F_m \in \{L_m^\pm, M_m^\pm\}$ does not change the results.

We again count how many highest weight vectors $g_k F$, $F = \prod_{s=1}^m F_s$, with weight $(k)_+'$ we find: for each $F_{2a-1} F_{2a}$, $1 \leq a \leq n$, we have 8 possible combinations, namely

$$\{L_{2a-1}^+ L_{2a}^+, L_{2a-1}^- L_{2a}^-, L_{2a-1}^+ M_{2a}^+, L_{2a-1}^- M_{2a}^-, M_{2a-1}^+ L_{2a}^+, M_{2a-1}^- M_{2a}^+, M_{2a-1}^- L_{2a}^-, M_{2a-1}^+ M_{2a}^-\},$$

and for F_m we have four possible choices L_m^\pm, M_m^\pm . Combining this, we find $8^n 2^2 = 2^{3n+2} = 2^{2m-n}$ isomorphic irreducible representations with highest weight $(k)_+'$, each of which has dimension

$$2^n \binom{k+m-2}{k}.$$

Hence the total dimension of all isomorphic irreducible representations is

$$2^{4n+2} \binom{k+m-2}{k} = \dim_{\mathbb{C}} \mathcal{M}_k,$$

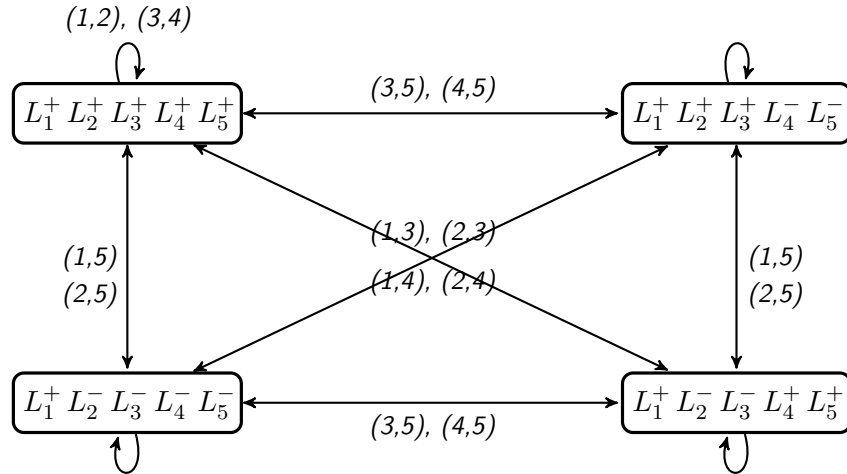
i.e. the dimensional analysis shows that \mathcal{M}_k may be decomposed as 2^{3n+2} isomorphic irreducible representations with highest weight $(k)_+'$.

Definition 3. We define the spinorspace \mathbb{S}_{2n+1} as the image under $\mathfrak{so}(m, \mathbb{C})$ of the idempotent $\prod_{s=1}^m L_s^+$:

$$\mathbb{S}_{2n+1} = \mathfrak{so}(m, \mathbb{C}) \left(\text{span}_{\mathbb{C}} \{L_1^+ L_2^+ \dots L_{2n}^+ L_{2n+1}^+\} \right).$$

The element $L_1^+ L_2^+ \dots L_{2n}^+ L_{2n+1}^+$ is a highest weight vector with weight $(0)_+ = (\frac{1}{2}, \dots, \frac{1}{2})$ and thus generates an irreducible representation with the same weight.

Example 2. Let $m = 5$ (i.e. $n = 2$) and consider $L = L_1^+ L_2^+ L_3^+ L_4^+ L_5^+$. We denote $dR(e_{a,b})$ in short as (a,b) . The Lie algebra $\mathfrak{so}(5, \mathbb{C})$ is given in this context by the span over \mathbb{C} of the ten elements $(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5)$ and $(4,5)$. The idempotents involved interact in the following way under the action of $\mathfrak{so}(m, \mathbb{C})$:



Hence

$$\mathbb{S}_5 = \text{span}_{\mathbb{C}} \{L_1^+ L_2^+ L_3^+ L_4^+ L_5^+, L_1^+ L_2^- L_3^- L_4^+ L_5^+, L_1^+ L_2^- L_3^- L_4^- L_5^-, L_1^+ L_2^+ L_3^+ L_4^- L_5^-\}.$$

The spinorspace \mathbb{S}_5 is 2^2 -dimensional.

In general, the rotations $dR(e_{2a-1,2a})$, $a = 1, \dots, n$, acting on an idempotent return the same idempotent up to a multiplicative complex factor. Again, we find that $dR(e_{a,b})$ with $1 \leq a, b \leq n$, changes the sign of an even number of L_i 's. The additional rotations $dR(e_{2a-1,m})$ and $dR(e_{2a,m})$, with $1 \leq a \leq n$, act as follows on L :

$$L_1^+ L_2^+ \dots L_{2n}^+ L_{2n+1}^+ \mapsto L_1^+ L_2^+ \dots L_{2a-1}^+ L_{2a}^- \dots L_{2n}^- L_{2n+1}^-.$$

The rotation always leaves L_1^+ invariant. The resulting idempotent will always have an even number of minus-signs. Starting from the idempotent L with all plus-signs, we thus get all possible idempotents of the following form:

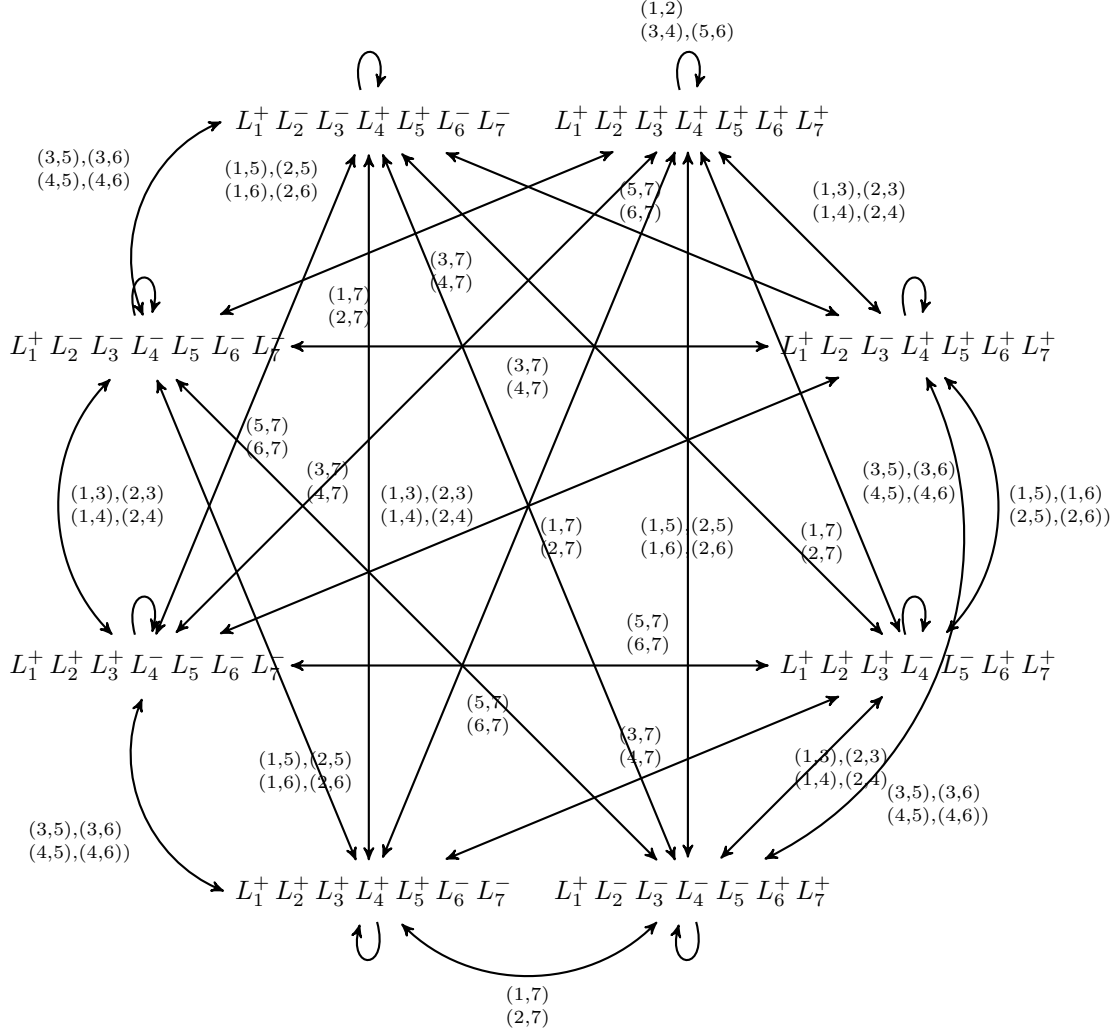
$$L_1^+ \underbrace{\quad} \underbrace{\quad} \dots \underbrace{\quad}$$

where each place $\underbrace{\quad}$ consists of either $L_{2a}^+ L_{2a+1}^+$ or $L_{2a}^- L_{2a+1}^-$, $1 \leq a \leq n$. We thus get 2^n spinors and we have the following weight space decomposition

$$\mathbb{S}_{2n+1} = \bigoplus V_{(\pm\frac{1}{2}, \pm\frac{1}{2}, \dots, \pm\frac{1}{2})},$$

where the sum goes over all weights with an even number of minus-signs.

Example 3. Let $m = 7$ (i.e. $n = 3$) and consider $L = L_1^+ L_2^+ L_3^+ L_4^+ L_5^+ L_6^+ L_7^+$. We will again denote $dR(e_{a,b})$ in short as (a,b) . The Lie algebra $\mathfrak{so}(5, \mathbb{C})$ is given in this context by 21 elements and the corresponding spinorspace will be 8-dimensional. The idempotents involved interact in the following way under the action of $\mathfrak{so}(m, \mathbb{C})$:



Hence

$$\mathbb{S}_7 = \text{span}_{\mathbb{C}} \{ L_1^+ L_2^+ L_3^+ L_4^+ L_5^+ L_6^+ L_7^+, L_1^+ L_2^- L_3^- L_4^+ L_5^+ L_6^- L_7^-, L_1^+ L_2^+ L_3^+ L_4^- L_5^- L_6^+ L_7^+, L_1^+ L_2^- L_3^- L_4^- L_5^- L_6^- L_7^-, L_1^+ L_2^+ L_3^+ L_4^+ L_5^- L_6^- L_7^-, L_1^+ L_2^- L_3^- L_4^- L_5^+ L_6^+ L_7^+, L_1^+ L_2^- L_3^- L_4^- L_5^- L_6^- L_7^-, L_1^+ L_2^+ L_3^+ L_4^+ L_5^+ L_6^+ L_7^+ \}.$$

We indeed find an 8-dimensional spinorspace \mathbb{S}_7 .

5 Conclusion and future research

The space \mathcal{M}_k of discrete k -homogeneous monogenic polynomials is a reducible representation of $\mathfrak{so}(m, \mathbb{C})$, which can, in the odd-dimensional case $m = 2n + 1$, be decomposed

into 2^{2m-n} isomorphic copies of the irreducible $\mathfrak{so}(m, \mathbb{C})$ -representation with highest weight $(k + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ and in the even-dimensional setting $m = 2n$, we find 2^{2m-n} isomorphic irreducible representations with highest weight $(k + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ combined with 2^{2m-n} irreps of highest weight $(k + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$. This is done by means of an appropriate amount of idempotents.

Let $g_k = (\xi_2 - \xi_1)(\xi_2 + \xi_1)(\xi_2 - \xi_1)(\xi_2 + \xi_1) \dots [1]$, (k factors), be a discrete homogeneous monogenic function of degree k and let

$$\begin{aligned} L_{2a-1}^\pm &= (\mathbf{e}_{2a-1}^+ \mathbf{e}_{2a-1}^- \pm i \mathbf{e}_{2a-1}^+), & L_{2a}^\pm &= (\mathbf{e}_{2a}^+ \mathbf{e}_{2a}^- \pm \mathbf{e}_{2a}^+), \\ M_{2a-1}^\pm &= (\mathbf{e}_{2a-1}^- \mathbf{e}_{2a-1}^+ \pm i \mathbf{e}_{2a-1}^-), & M_{2a}^\pm &= (\mathbf{e}_{2a}^- \mathbf{e}_{2a}^+ \pm \mathbf{e}_{2a}^-), \end{aligned}$$

Denote $\|L_a^\pm\| = 0$, $\|M_a^\pm\| = 1$ and $|L_a^+| = |M_a^-| = 0$ and $|L_a^-| = |M_a^+| = 1$.

In even dimension $m = 2n$, the polynomial $g_k F \in \mathcal{M}_k$, $F = \prod_{s=1}^m F_s$ with $F_s \in \{L_s^\pm, M_s^\pm\}$, is a weight vector of $\mathfrak{so}(m, \mathbb{C})$ with

- weight $(k)_+'$ when $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ is even and $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ is even for $2 \leq a \leq n$.
- weight $(k)_-'$ when $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ is even, $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ is even for $2 \leq a \leq n-1$ and $\|F_{2n-1}\| + \|F_{2n}\| + |F_{2n-1}| + |F_{2n}|$ is odd.

We find 2^{2m-n} highest weight vectors in \mathcal{M}_k with weight $(k)_+'$ and 2^{2m-n} weight vectors, with weight $(k)_-'$.

In odd dimensions $m = 2n + 1$, the polynomial $g_k F \in \mathcal{M}_k$, $F = \prod_{s=1}^m F_s$ with $F_s \in \{L_s^\pm, M_s^\pm\}$, is a weight vector of $\mathfrak{so}(m, \mathbb{C})$ with weight $(k)_+'$ when $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ is even and $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ is even for $2 \leq a \leq n$. We find 2^{2m-n} highest weight vectors in \mathcal{M}_k with weight $(k)_+'$.

We have proven throughout this article how the spaces \mathcal{H}_k and \mathcal{M}_k of harmonic resp. monogenic discrete k -homogeneous polynomials may be decomposed into irreducible representations of $\mathfrak{so}(m, \mathbb{C})$. However, because of the presence of the basiselements e_a and e_a^\perp in the definition of the generators of the rotations, the spinorspace is no maximal left ideal. In future research we will investigate other possibilities to define rotations and the spinorspace in the hopes of writing the spinorspace as maximal left ideal. An equivalent description of \mathcal{H}_k and \mathcal{M}_k as $SO(m)$ -representations is also still work in progress.

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